DE LA RECHERCHE À L'INDUSTRIE

Positive schemes for diffusion problems on deformed meshes

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december 8-10, 2015

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Outline

Context

Rosseland approximation hydro-rad system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0^a, \\ \partial_t (\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla(p + p_r) = \mathbf{0}^b, \\ \partial_t(\rho e + e_r) + \operatorname{div}((\rho e + e_r + p + p_r)\mathbf{u}) = \operatorname{div}\left(\dfrac{c}{3\sigma_R}\nabla e_r\right) + S^c. \end{cases}$$

[a] ρ the density, $\mathbf{u}$ the fluid velocity.

[b] p the material pressure, $p_r = 1/3aT^4$ the radiative pressure, T the temperature.

[c] $e = \varepsilon + 1/2\mathbf{u}^2$, the total material energy, $e_r = aT^4$ the radiative energy, c the speed of light, $\sigma_R(T)$ the Rosseland opacity, $\varepsilon(\rho, T)$ the internal energy and S the energy source.

Mathematical validity domain of this system

$\rho > 0$, $T > 0$.

Solving strategy and requirements

Splitting

- Splitting into a hydrodynamic part and a radiative part.
- Godunov-type Lagrangian scheme Glace [a] for the hydrodynamic part.

[a] B. Després and C. Mazeran *Arch. Rat. Mech. anal.* 2005

We focus on the radiative part

$$\partial_t(\rho\varepsilon(T) + aT^4) = \operatorname{div}\left(\frac{c}{3\sigma_R}\nabla(aT^4)\right) + S.$$

Non-linear parabolic equation on T.

$$\boxed{(\partial_T\varepsilon > 0, \partial_T e_r > 0, \sigma_R > 0, T|_\Gamma > 0, S > 0) \Rightarrow T > 0.}$$

To conserve this property at the discrete level: need of positive (monotone) diffusion scheme.

Some works on non-linear positive diffusion

- E. Bertolazzi et G. Manzini, A second-order maximum principle preserving finite volume method for steady convection-diffusion problems, SIAM, 2005.
- J. Droniou et C. Le Potier, Construction and convergence study of local-maximum-principle preserving schemes for elliptic equations, SIAM Journal on Numerical Analysis, 2011.
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[1] Local-maximum-principle enforcement

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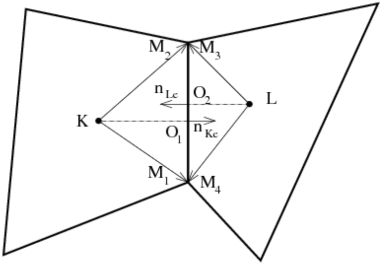
Benchmarks (Samuel De Santis training 2010):

pro LMP[1] ⇒ positivity.

con LMP ⇒ more constraints on the non-linear system.

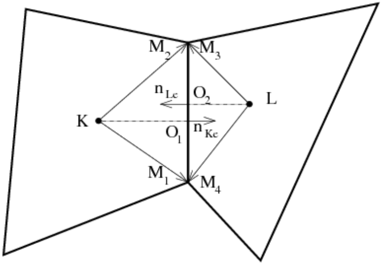
[1] Local-maximum-principle enforcement

Building a positive scheme



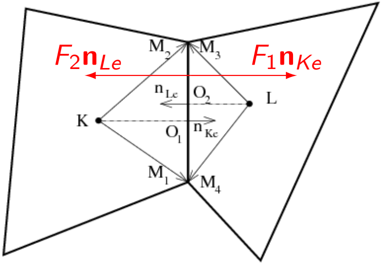
$$\int_K \frac{\partial u}{\partial t} dx - \int_K \nabla \cdot (\kappa \nabla u) dx = \int_K f dx,$$

Building a positive scheme



$$\int_K \frac{\partial u}{\partial t} dx + \sum_{e \in \partial K} \underbrace{\left(- \int_e \kappa \nabla u \cdot \mathbf{n}_{K_e} d\Gamma \right)}_{\mathcal{F}_{K,e}} = \int_K f dx.$$

Building a positive scheme



$$F_1 = \mathcal{F}_{K,e} + O(\Delta x^2)$$

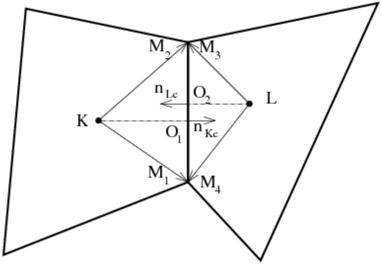
$$F_2 = \mathcal{F}_{L,e} + O(\Delta x^2)$$

$$\mu_1 F_1 - \mu_2 F_2 = \mathcal{F}_{K,e} + O(\Delta x^2)$$
$$= -\mathcal{F}_{L,e} + O(\Delta x^2)$$

$$\forall \{\mu_1 \geq 0, \mu_2 \geq 0\}, \mu_1 + \mu_2 = 1$$

$$\int_K \frac{\partial u}{\partial t} dx + \sum_{e \in \partial K} \underbrace{\left(- \int_e \kappa \nabla u \cdot \mathbf{n}_{K_e} d\Gamma \right)}_{\mathcal{F}_{K,e}} = \int_K f dx.$$

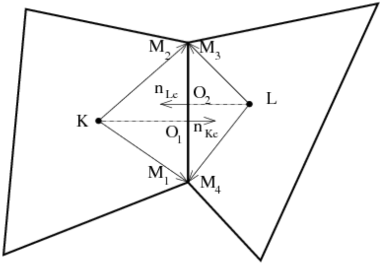
Building F_1 and F_2



$$\mathcal{F}_{Ke} = -\int_e \kappa \nabla u \cdot \mathbf{n}_{K_e} d\Gamma.$$

- Find $\{M_1, M_2, M_3, M_4\}$, s.t. $\mathbf{n}_{Ke} = \lambda_1 \mathbf{KM}_1 + \lambda_2 \mathbf{KM}_2$, $\mathbf{n}_{Le} = \lambda_3 \mathbf{KM}_3 + \lambda_4 \mathbf{KM}_4$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$.
- $F_1 = -|e|\kappa_e \left(\lambda_1(u_{M_1} - u_K) + \lambda_2(u_{M_2} - u_K)\right)$
- $F_2 = -|e|\kappa_e \left(\lambda_3(u_{M_3} - u_L) + \lambda_4(u_{M_4} - u_L)\right)$
- $F_{Ke} = \mu_1 F_1 - \mu_2 F_2 = \mathcal{F}_{Ke} + O(\Delta x^2)$

Calculation of μ_1 and μ_2

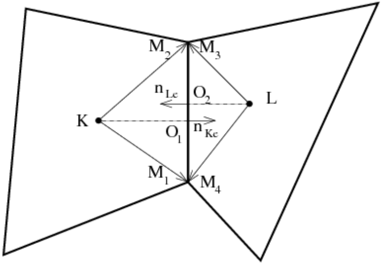


$$F_{Ke} = \mu_1 F_1 - \mu_2 F_2$$

$$F_{Ke} = |e|\kappa_e \Big(\mu_1(\lambda_1+\lambda_2)u_K - \mu_2(\lambda_3+\lambda_4)u_L - \mu_1 \underbrace{(\lambda_1 u_{M_1} + \lambda_2 u_{M_2})}_{a_1} + \mu_2 \underbrace{(\lambda_3 u_{M_3} + \lambda_4 u_{M_4})}_{a_2}\Big)$$

$$\begin{cases} \mu_1 + \mu_2 & = 1 \\ a_1\mu_1 - a_2\mu_2 & = 0 \end{cases}$$

Calculation of μ_1 and μ_2



$$F_{Ke} = \mu_1 F_1 - \mu_2 F_2$$

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$$\begin{cases} \mu_1 + \mu_2 & = 1 \\ a_1\mu_1 - a_2\mu_2 & = 0 \end{cases}$$

Calculation of μ_1 and μ_2 - continued

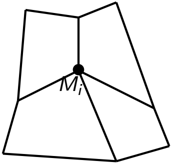
We obtain

$$F_{Ke} = |e|\kappa_e(\mu_1(\lambda_1 + \lambda_2)u_K - \mu_2(\lambda_3 + \lambda_4)u_L)$$

with: $$\begin{cases} \mu_1 = \dfrac{a_2}{a_1 + a_2}, \mu_2 = \dfrac{a_1}{a_1 + a_2}, \\ a_1 = \lambda_1 u_{M_1} + \lambda_2 u_{M_2}, a_2 = \lambda_3 u_{M_3} + \lambda_4 u_{M_4} \end{cases}$$

- The scheme is non-linear,
- the scheme is positive iff $a_1 \geq 0$ and $a_2 \geq 0$.
- a sufficient condition for that is $u_{M_1}, u_{M_2}, u_{M_3}, u_{M_4} \geq 0$,
- the matrix obtained is non-symmetric,
- degenerate on TPFA scheme on cartesian grids.

Secondary unknowns u_{M_i} issue



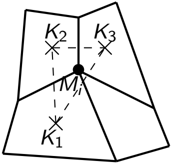
Sheng, Yuan and Yue initial solution

Find three cell-centers geometrically surrounding M_i and perform a barycentric interpolation of u_{M_i}.

- pro Second-order accuracy,
- pro positive weights,
- con If the mesh is heavily deformed, centers can be far from M_i (parallelism issue) and even not exist.
- con Non unique.

ABANDONNED...

Secondary unknowns u_{M_i} issue



$$u_{M_i} = \omega_1 u_{K_1} + \omega_2 u_{K_2} + \omega_3 u_{K_3}$$
$$\omega_1, \omega_2, \omega_3 \geq 0$$

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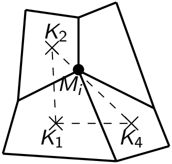

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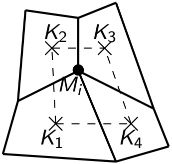


$$u_{M_i} = \omega_1 u_{K_1} + \omega_2 u_{K_2} + \omega_3 u_{K_3} + \omega_4 u_{K_4}$$
$$\omega_1, \omega_2, \omega_3, \omega_4 \geq 0$$

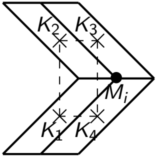
Sheng, Yuan, Yue Modified (SYYM) Xavier Blanc, E. L., ZAMM 2015

Use all cell-centers topologically surrounding M_i and perform a least-squares interpolation of u_{M_i}.

- pro Second-order accuracy,
- pro easy parallelism,
- con If the mesh is heavily deformed, some weights can be negative.

Truncate negative u_{M_i} to zero (legal because we assume $u > 0$).

Secondary unknowns u_{M_i} issue



$$u_{M_i} = \max(0, \omega_1 u_{K_1} + \omega_2 u_{K_2} + \omega_3 u_{K_3} + \omega_4 u_{K_4})$$
$$\text{some } \omega_i \leq 0$$

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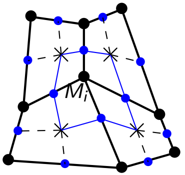


$$\int \kappa \nabla u \cdot \mathbf{n} d\Gamma$$

approximated on the dual (blue) cell
with the algorithm described before.

DDFV-like strategy , J.-S. Camier, F. Hermeline, IJNM Engineering, to appear

Solve a second diffusion problem on the dual mesh to compute M_i.

- pro Second-order accuracy,
- pro easy parallelism,
- con number of degrees of freedom multiplied by two.

Called NLMDDFV in the sequel.

General features of the scheme(s)

Properties

- Positive (for converged linear solver),
- conservative,
- non-linear (fixed point strategy required even for linear problems),
- existence of a solution,
- convergence to the solution under a parabolic CFL condition.

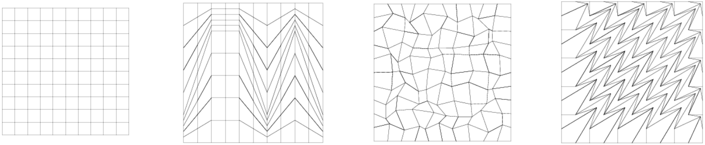
Associated matrices (backward Euler in time)

- Non-symmetric, M^T is a M-matrix,
- very sparse (only 5 non zero coefficients per row in 2D on structured square mesh).

Steady analytic problem

Initial and boundary conditions

- $-\Delta u = 2\pi^2 \sin(\pi x) sin(\pi y), u|_\Gamma = 0, \Omega = [0,1]^2$,
- mesh convergency from 10^2 to 1280^2 zones,
- four mesh patterns.



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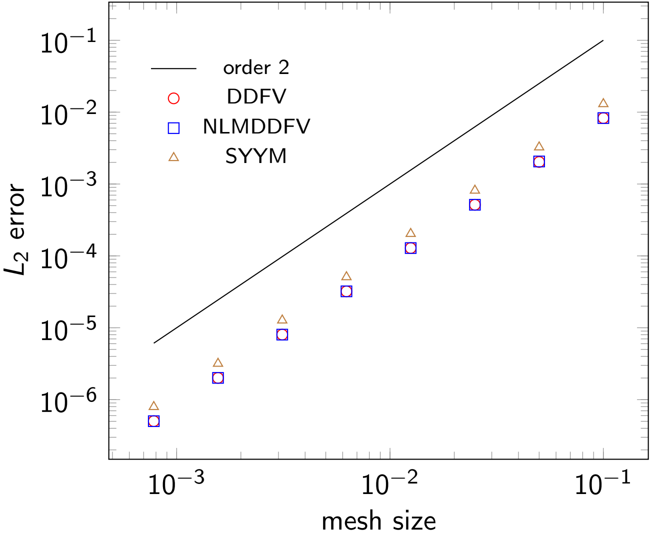

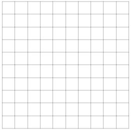

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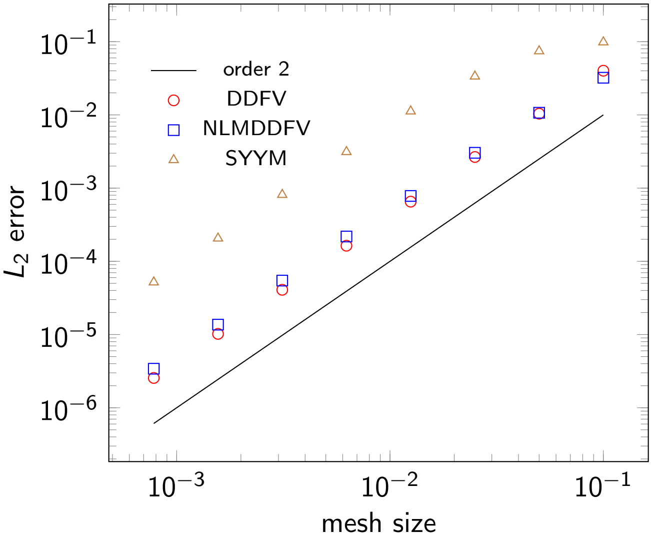

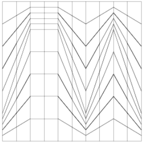

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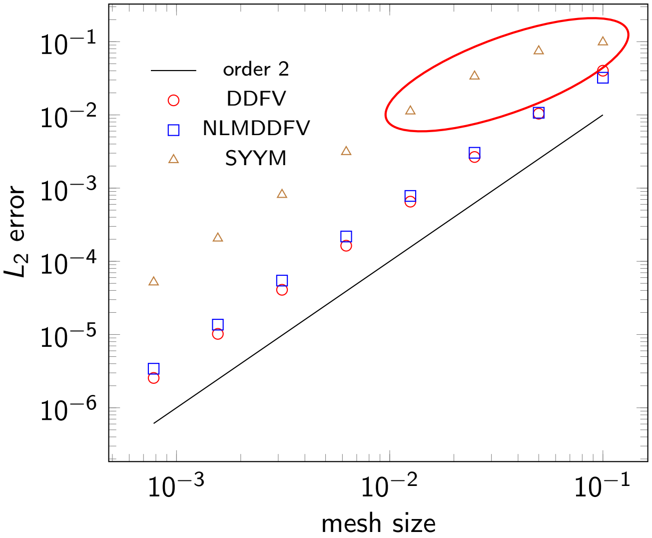

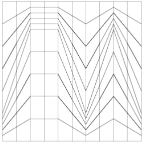

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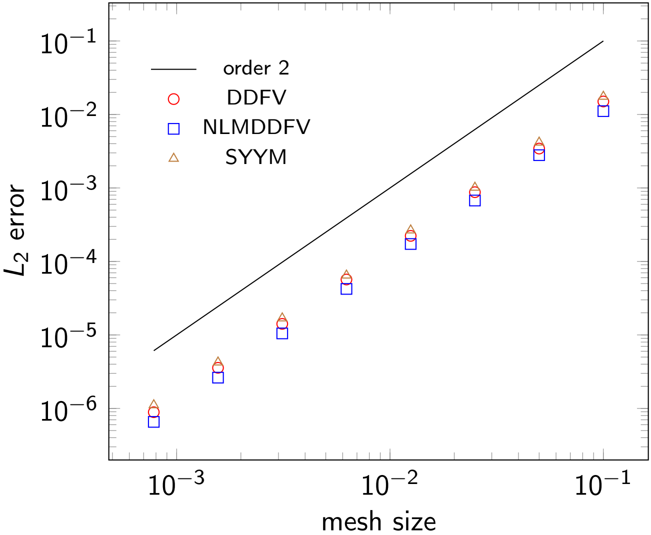

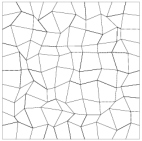

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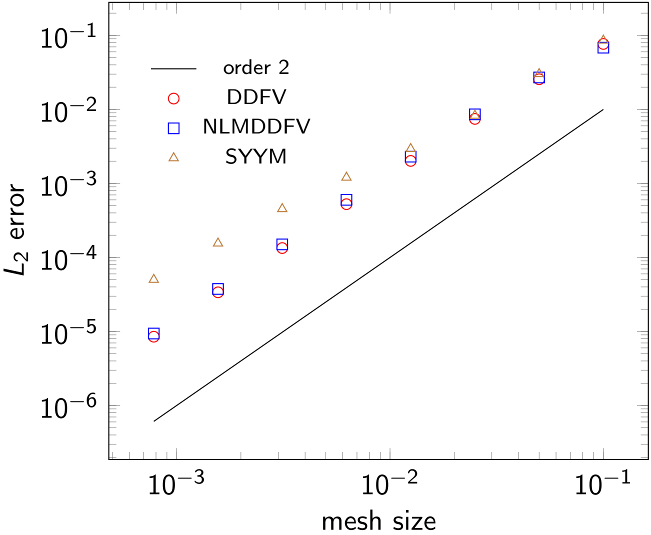

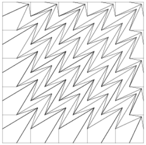

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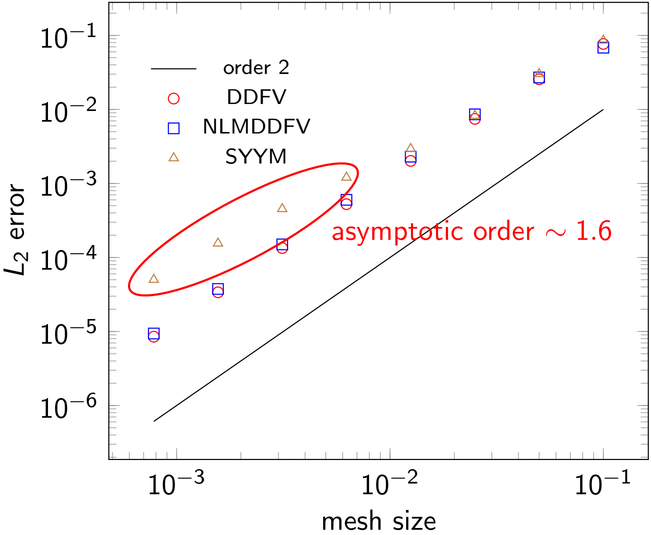
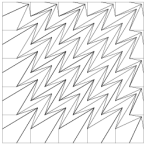
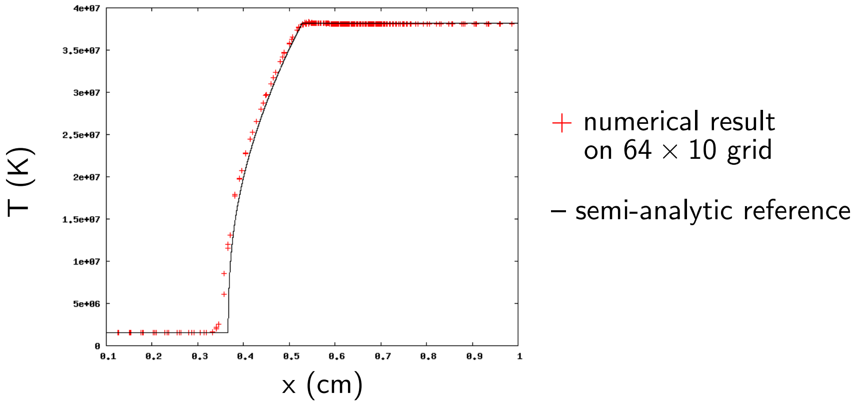




Semi-analytic radiative shock

1D hydro-radiative problem with semi-analytic solution[2] computed on kershaw mesh. Fails with mimetic schemes without fix.



[2] R. Lowrie and R. Rauenzahn, Shock Waves, 2007

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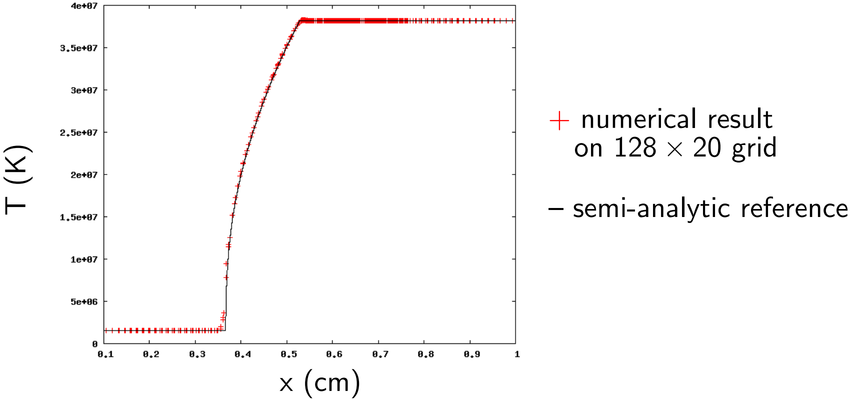

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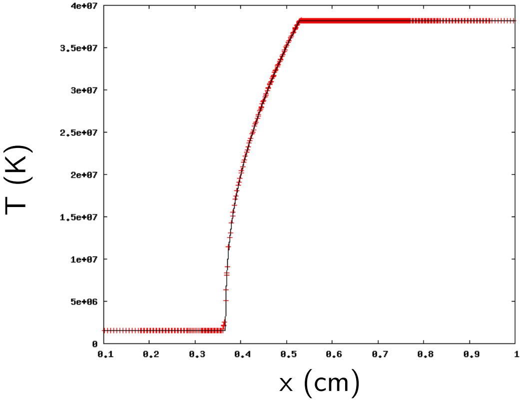

+ numerical result on 256×40 grid

– semi-analytic reference

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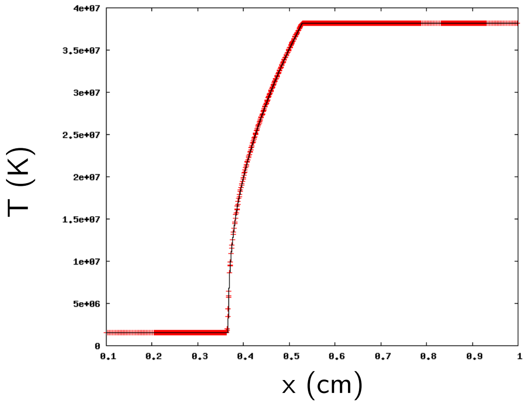


\+ numerical result on 512×80 grid

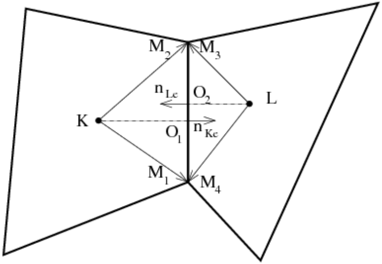
– semi-analytic reference

[2]R. Lowrie and R. Rauenzahn, Shock Waves, 2007

Final remark

This kind of schemes widely increases the robustness of our calculations on deformed meshes.

Building F_1 and F_2



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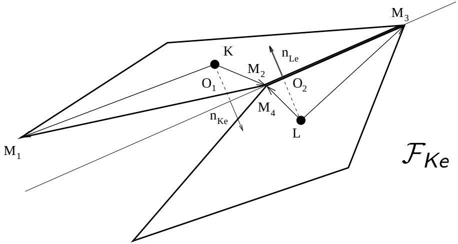

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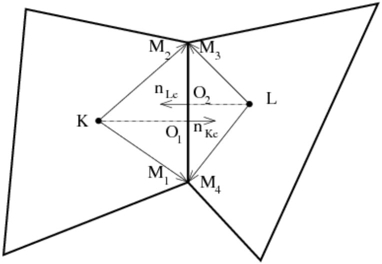
Calculation of μ_1 and μ_2



$$F_{Ke} = \mu_1 F_1 - \mu_2 F_2$$

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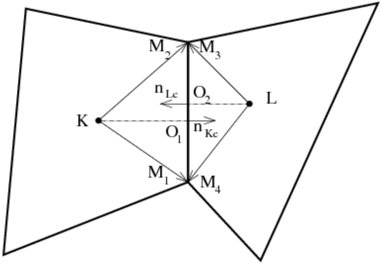
$$\begin{cases} \mu_1 + \mu_2 & = 1 \\ a_1\mu_1 - a_2\mu_2 & = 0 \end{cases}$$

Calculation of μ_1 and μ_2

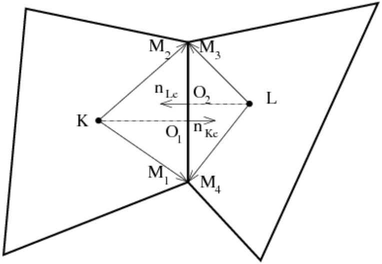


$$F_{Ke} = \mu_1 F_1 - \mu_2 F_2$$

$$F_{Ke} = |e| \kappa_e \quad (\mu_1(\lambda_1 + \lambda_2) u_K - \mu_2(\lambda_3 + \lambda_4) u_L)$$

$$\begin{cases} \mu_1 + \mu_2 & = 1 \\ a_1 \mu_1 - a_2 \mu_2 & = 0 \end{cases}$$

Calculation of μ_1 and μ_2



$$F_{Ke} = \mu_1 F_1 - \mu_2 F_2$$

$$F_{Ke} = |e|\kappa_e \quad (\ \mu_1(\lambda_1 + \lambda_2)u_K - \mu_2(\lambda_3 + \lambda_4)u_L)$$

$$\begin{cases} \mu_1 + \mu_2 & = 1 \\ (a_1 + \Delta x^2)\mu_1 - (a_2 + \Delta x^2)\mu_2 & = 0 \end{cases}$$

Calculation of μ_1 and μ_2 - continued

We obtain

$$F_{Ke} = |e|\kappa_e(\mu_1(\lambda_1+\lambda_2)u_K - \mu_2(\lambda_3+\lambda_4)u_L)$$

with: $$\begin{cases} \mu_1 = \dfrac{a_2+\Delta x^2}{a_1+a_2+2\Delta x^2}, \mu_2 = \dfrac{a_1+\Delta x^2}{a_1+a_2+2\Delta x^2}, \\ \\ a_1 = \lambda_1 u_{M_1} + \lambda_2 u_{M_2}, a_2 = \lambda_3 u_{M_3} + \lambda_4 u_{M_4} \end{cases}$$

- The scheme is non-linear,
- the scheme is positive iff $a_1 \geq 0$ and $a_2 \geq 0$.
- a sufficient condition for that is $u_{M_1}, u_{M_2}, u_{M_3}, u_{M_4} \geq 0$,
- the matrix obtained is non-symmetric,
- degenerate on TPFA scheme on cartesian grids.